

Computations with nilpotent orbits in SLA

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Abstract

We report on some computations with nilpotent orbits in simple Lie algebras of exceptional type within the **SLA** package of **GAP4**. Concerning reachable nilpotent orbits our computations firstly confirm the classification of such orbits in Lie algebras of exceptional type by Elashvili and Grélaud, secondly they answer a question by Panyushev, and thirdly they show in what way a recent result of Yakimova for the Lie algebras of classical type extends to the exceptional types. The second topic of this note concerns abelianizations of centralizers of nilpotent elements. We give tables with their dimensions.

1 Introduction

In the theory of the nilpotent orbits of simple Lie algebras there appears to be a dichotomy between the Lie algebras of classical type and those of exceptional type: statements for the nilpotent orbits in the Lie algebras of classical type usually have an elegant proof, whereas the same statements for the exceptional types require detailed and “dirty” calculations (with or without making use of a computer). This can already be seen in the classification of the nilpotent orbits itself: for the classical types they are classified in terms of partitions that directly correspond to the Jordan blocks of a representative, while for the exceptional types completely different methods have to be used. Another example is the proof of Elashvili’s conjecture: this was proved for the classical types by Yakimova ([15]) and checked by computer for the exceptional types in [10]. Leter Charbonnel and Moreau found a more uniform proof ([4]), which, however, for some cases in the exceptional types still required computer calculations.

This paper is devoted to some more instances of this pattern. In the first half we present some results, obtained by computer calculations, on *reachable* nilpotent orbits of simple Lie algebras of exceptional type. They extend results obtained for the classical types ([13], [16]). In the second half we present computational data on the algebras $\mathfrak{g}_e/[\mathfrak{g}_e, \mathfrak{g}_e]$. This data has been used by Premet and Topley ([14]) to extend some of their results, related to finite W -algebras, for the classical types to the exceptional types. (Here the

use of “extend” includes the possibility that the statements are slightly changed, or that exceptions are introduced.)

All results given here have been obtained by computation - in particular no proofs are given (the algorithms used have been described elsewhere, cf. [10]). This paper is intended to serve two purposes: firstly to publish the results of our computations for future reference, and secondly to advertise the package **SLA** with which the computations have been done.

1.1 Preliminaries on nilpotent orbits

Now we introduce some notation and recall some definitions and facts on nilpotent orbits. For more background information we refer to the book by Collingwood and McGovern ([5]).

Let \mathfrak{g} be a simple Lie algebra over \mathbb{C} (or over an algebraically closed field of characteristic 0). Let G denote the adjoint group of \mathfrak{g} . By the Jacobson-Morozov theorem a nilpotent $e \in \mathfrak{g}$ lies in an \mathfrak{sl}_2 -triple (h, e, f) (where $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$). Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra containing h . Let Φ be the root system of \mathfrak{g} with respect to \mathfrak{h} , with basis of simple roots $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. Let W denote the Weyl group of Φ . Then, possibly after replacing h with a W -conjugate, we have $\alpha_i(h) \in \{0, 1, 2\}$. The Dynkin diagram of Δ , where the node corresponding to α_i is labeled $\alpha_i(h)$, is called a weighted Dynkin diagram. It uniquely determines the orbit Ge .

For $e \in \mathfrak{g}$ we denote its centraliser in \mathfrak{g} by \mathfrak{g}_e . In [13] an e in \mathfrak{g} is defined to be *reachable* if $e \in [\mathfrak{g}_e, \mathfrak{g}_e]$. Such an element has to be nilpotent. In [7], Elashvili and Grélaud gave a classification of reachable elements in \mathfrak{g} (in that paper such elements are called *compact*, in analogy with [1]).

By the adjoint representation the subalgebra spanned by an \mathfrak{sl}_2 -triple (h, e, f) acts on \mathfrak{g} . Since the eigenvalues of $\text{ad}h$ are integers, we get a grading

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}(k)$$

where $\mathfrak{g}(k) = \{x \in \mathfrak{g} \mid [h, x] = kx\}$. Now set $\mathfrak{g}(k)_e = \mathfrak{g}(k) \cap \mathfrak{g}_e$, and let $\mathfrak{g}(\geq 1)_e$ denote the subalgebra spanned by all $\mathfrak{g}(k)_e$, $k \geq 1$.

Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra, with Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$, where \mathfrak{n} is the nilradical, and \mathfrak{l} is reductive. Let $L \subset G$ be the connected subgroup of G with Lie algebra \mathfrak{l} . Let Le' be a nilpotent orbit in \mathfrak{l} . Then Lusztig and Spaltenstein ([12]) have shown that there is a unique nilpotent orbit $Ge \subset \mathfrak{g}$ such that $Ge \cap (Le' \oplus \mathfrak{n})$ is dense in the latter. The orbit Ge is said to be *induced* from the orbit Le' . Nilpotent orbits which are not induced are called *rigid*.

Let n be a non-negative integer. The irreducible components of the locally closed set

$$A^n = \{x \in \mathfrak{g} \mid \dim Gx = n\}$$

are called *sheets* of \mathfrak{g} (see [2], [3]). A sheet is a G -stable subset containing a *unique* nilpotent orbit. The converse is not true: sheets in general are not disjoint, and different sheets may

contain the same nilpotent orbit. The sheets of \mathfrak{g} are in bijection with (G -classes of) pairs (\mathfrak{l}, Le') , where \mathfrak{l} is a Levi subalgebra, and Le' is a rigid nilpotent orbit in \mathfrak{l} . The *rank* of the sheet corresponding to the pair (\mathfrak{l}, Le') is defined to be the dimension of the centre of \mathfrak{l} .

1.2 Main results

Panyushev ([13]) showed that, for \mathfrak{g} of type A_n , e is reachable if and only if $\mathfrak{g}(\geq 1)_e$ is generated as Lie algebra by $\mathfrak{g}(1)_e$. Here we call this the *Panyushev property* of \mathfrak{g} . In [13] it is stated that this property also holds for the other classical types and the question is posed whether it holds for the exceptional types. In [16] a proof is given that the Panyushev property holds in types B_n, C_n, D_n . Our computations confirm that the Panyushev property holds also for the Lie algebras of exceptional type.

Yakimova ([16]) studied the stronger condition $\mathfrak{g}_e = [\mathfrak{g}_e, \mathfrak{g}_e]$. For the purposes of this paper we call elements e satisfying this condition *strongly reachable*. She showed that for \mathfrak{g} of classical type, e is strongly reachable if and only if the nilpotent orbit of e is rigid. Furthermore, this is shown to fail for \mathfrak{g} of exceptional type. As a result of our calculations (Section 2) we find all rigid nilpotent orbits whose representatives are not strongly reachable. From this we conclude that e is strongly reachable if and only if e is both reachable and rigid. We note that one direction of this statement can be shown in a uniform way for all \mathfrak{g} : if e is strongly reachable then it is reachable, but also rigid by [16], Proposition 11. The converse for exceptional types follows from our calculations in two ways. Firstly we compute the list of all strongly reachable orbits and the list of all nilpotent orbits that are reachable and rigid, and find that they are the same. Second, the Panyushev property, which we checked by computation for the exceptional types, also implies the statement. (For the classical types we have, of course, the stronger theorem from [16].)

Let $e \in \mathfrak{g}$ be a nilpotent element lying in an \mathfrak{sl}_2 -triple (h, e, f) . In Section 3 we consider the quotient $\mathfrak{c}_e = \mathfrak{g}_e / [\mathfrak{g}_e, \mathfrak{g}_e]$, on which $\text{ad}h$ acts with non-negative eigenvalues. For each \mathfrak{g} of exceptional type, we give a table listing the dimension of \mathfrak{c}_e , where e runs over a set of representatives of the nilpotent orbits in \mathfrak{g} , as well as the eigenvalues of $\text{ad}h$, acting on \mathfrak{c}_e , with their multiplicities. Among other things, these tables show that if Ge is an induced nilpotent orbit that lies in a unique sheet, then the rank of that sheet almost always equals $\dim \mathfrak{c}_e$. The six exceptions are explicitly listed (Proposition 2). Premet and Topley ([14]) proved that this holds without exceptions for the classical Lie algebras. Proposition 2, as well as the tables of Section 3, are used in [14] for showing that $U(\mathfrak{g}, e)^{\text{ab}}$ (the abelianization of a finite W -algebra $U(\mathfrak{g}, e)$) is isomorphic to a polynomial ring (with the same six exceptions as Proposition 2).

1.3 The SLA package

The SLA package ([9] - the acronym stands for Simple Lie Algebras), is written in the language of the computer algebra system GAP4 ([8]), and it can be freely downloaded. As the name indicates it has functionality for working with (semi-) simple Lie algebras.

Currently there are three main areas which are touched upon by the package: nilpotent orbits in semisimple Lie algebras, nilpotent orbits of θ -groups, and semisimple subalgebras of semisimple Lie algebras.

For the computations underpinning the results of this paper we have used the functionality for the nilpotent orbits in simple Lie algebras. In particular the package contains the classification of such orbits. Using this it is straightforward to approach the above questions by computational means. Indeed, for a nilpotent orbit the system easily computes a representative e , and a corresponding \mathfrak{sl}_2 -triple (h, e, f) . Then using functions present in **GAP4** we can compute the centralizer, \mathfrak{g}_e , and its derived subalgebra, and check whether e lies in it. This gives us the list of reachable nilpotent orbits. Secondly, a similar procedure yields the list of strongly reachable orbits. Thirdly, **SLA** has a function for computing the grading corresponding to an \mathfrak{sl}_2 -triple. With that it is straightforward to check whether $\mathfrak{g}(\geq 1)_e$ is generated by $\mathfrak{g}(1)_e$. Finally, a small procedure for constructing the space $\mathfrak{c}_e = \mathfrak{g}_e/[\mathfrak{g}_e, \mathfrak{g}_e]$ and the action of h on it, is easily written. Using that the results of Section 3 are obtained.

Acknowledgement: I thank Alexander Elashvili for suggesting the topic of reachable nilpotent orbits. I thank Alexander Premet for suggesting the computations relative to Section 3.

2 Reachable nilpotent elements in the Lie algebras of exceptional type

Tables 1, 2, 3, 4, and 5 contain the nilpotent orbits that by our calculations are reachable. The content of the tables is as follows. The first column has the label of the orbit, and the second column the weighted Dynkin diagram. The third and fourth columns contain a \times if the orbit is, respectively, strongly reachable and rigid. We note that the classification of rigid nilpotent orbits is known (see [6], [11]).

Table 1: Reachable nilpotent orbits in E_6 .

label	weighted Dynkin diagram	Strong	Rigid
A_1	$\begin{array}{ccccc} & & 1 & & \\ & & 0 & & \\ 0 & 0 & 0 & 0 & 0 \end{array}$	\times	\times
$2A_1$	$\begin{array}{ccccc} & & 0 & & \\ & & 0 & & \\ 1 & 0 & 0 & 0 & 1 \end{array}$		
$3A_1$	$\begin{array}{ccccc} & & 0 & & \\ & & 1 & & \\ 0 & 0 & 1 & 0 & 0 \end{array}$	\times	\times
$A_2 + A_1$	$\begin{array}{ccccc} & & 1 & & \\ & & 0 & & \\ 1 & 0 & 0 & 0 & 1 \end{array}$		
$A_2 + 2A_1$	$\begin{array}{ccccc} & & 0 & & \\ & & 0 & & \\ 0 & 1 & 0 & 1 & 0 \end{array}$		
$2A_2 + A_1$	$\begin{array}{ccccc} & & 0 & & \\ & & 1 & & \\ 1 & 0 & 1 & 0 & 1 \end{array}$	\times	\times

Table 2: Reachable nilpotent orbits in E_7 .

label	weighted Dynkin diagram						Strong	Rigid
A_1	1	0	0	0	0	0	\times	\times
$2A_1$	0	0	0	0	1	0	\times	\times
$(3A_1)'$	0	1	0	0	0	0	\times	\times
$4A_1$	0	0	0	0	0	1	\times	\times
$A_2 + A_1$	1	0	0	0	1	0		
$A_2 + 2A_1$	0	0	1	0	0	0	\times	\times
$2A_2 + A_1$	0	1	0	0	1	0	\times	\times
$A_4 + A_1$	1	0	1	0	1	0		

 Table 3: Reachable nilpotent orbits in E_8 .

label	weighted Dynkin diagram							Strong	Rigid
A_1	0	0	0	0	0	0	1	\times	\times
$2A_1$	1	0	0	0	0	0	0	\times	\times
$3A_1$	0	0	0	0	0	1	0	\times	\times
$4A_1$	0	0	0	0	0	0	0	\times	\times
$A_2 + A_1$	1	0	0	0	0	0	1	\times	\times
$A_2 + 2A_1$	0	0	0	0	1	0	0	\times	\times
$A_2 + 3A_1$	0	1	0	0	0	0	0	\times	\times
$2A_2 + A_1$	1	0	0	0	0	1	0	\times	\times
$A_4 + A_1$	1	0	0	0	1	0	1		
$2A_2 + 2A_1$	0	0	0	1	0	0	0	\times	\times
$(A_3 + 2A_1)''$	0	1	0	0	0	0	1	\times	\times
$D_4(a_1) + A_1$	0	0	0	0	0	1	0	\times	\times
$A_3 + A_2 + A_1$	0	0	1	0	0	0	0	\times	\times
$2A_3$	1	0	0	1	0	0	0	\times	\times

<i>Reachable nilpotent orbits in E_8.</i>									
$A_4 + 2A_1$	0	0	0	1	0	0	0	1	
$A_4 + A_3$	0	0	0	1	0	0	1	0	\times

Table 4: Reachable nilpotent orbits in F_4 .

label	weighted Dynkin diagram	Strong	Rigid
A_1	1 0 0 0	\times	\times
\tilde{A}_1	0 0 0 1	\times	\times
$A_1 + \tilde{A}_1$	0 1 0 0	\times	\times
$A_2 + \tilde{A}_1$	0 0 1 0	\times	\times

Table 5: Reachable nilpotent orbits in G_2 .

label	weighted Dynkin diagram	Strong	Rigid
\tilde{A}_1	1 0	\times	\times

We make the following comments.

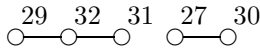
- Here the reachable elements are exactly the same as in the paper of Elashvili and Grélaud. Therefore our calculations confirm their result.
- The rigid nilpotent orbits that are not strongly reachable are
 - in type E_7 : $(A_3 + A_1)'$ (41, 40),
 - in type E_8 : $A_3 + A_1$ (84, 83), $D_5(a_1) + A_2$ (46, 45), $A_5 + A_1$ (46, 45),
 - in type F_4 : $\tilde{A}_2 + A_1$ (16, 15),
 - in type G_2 : A_1 (6, 5).

Here the pair of integers in brackets is $(\dim \mathfrak{g}_e, \dim[\mathfrak{g}_e, \mathfrak{g}_e])$.

- In type E_6 all rigid orbits are strongly reachable. Hence in this type the situation is the same as for the classical types: e is strongly reachable if and only if the orbit of e is rigid.
- The last two columns of all tables are equal. This shows that for the exceptional types the following theorem holds: e is strongly reachable if and only if e is both reachable and rigid.

- This last statement also follows from the Panyushev property. Indeed, e rigid implies that $\mathfrak{g}(0)_e$ is semisimple, so $[\mathfrak{g}(0)_e, \mathfrak{g}(0)_e] = \mathfrak{g}(0)_e$. Furthermore, $[\mathfrak{g}(0)_e, \mathfrak{g}(1)_e] = \mathfrak{g}(1)_e$ by [16], Lemma 8 (where this is shown to hold for all nilpotent e). By the Panyushev property this implies that $[\mathfrak{g}_e, \mathfrak{g}_e] = \mathfrak{g}_e$.
- We see that for all nilpotent orbits that are rigid but not strongly reachable the codimension of $[\mathfrak{g}_e, \mathfrak{g}_e]$ in \mathfrak{g}_e is 1. Since a rigid orbit is reachable if and only if it is strongly reachable, we get that in all those cases e spans the quotient $\mathfrak{g}_e/[\mathfrak{g}_e, \mathfrak{g}_e]$.

Example 1 Let us consider the nilpotent orbit in the Lie algebra of type E_7 with label $A_3 + A_2$. This orbit is not reachable. It has a representative with diagram



This means that the representative is $e = x_{29} + x_{32} + x_{31} + x_{27} + x_{30}$, where x_i denotes the root vector corresponding to the i -th positive root (enumeration as in GAP4, cf. [10]). Furthermore, the Dynkin diagram of these roots is as shown above. This representative is stored in the package SLA.

Now, if the orbit were reachable then $e \in [\mathfrak{g}_e, \mathfrak{g}_e] \cap \mathfrak{g}(2)$. Using the SLA package we can easily compute the latter space:

```
gap> L:= SimpleLieAlgebra("E",7,Rationals);;
gap> o:= NilpotentOrbits(L);;
gap> sl2:=SL2Triple( o[19] );
[ (2)*v.90+(3)*v.92+(2)*v.93+(3)*v.94+(4)*v.95, (6)*v.127+(9)*v.128+(12)*v.129
+(18)*v.130+(14)*v.131+(10)*v.132+(5)*v.133, v.27+v.29+v.30+v.31+v.32 ]
gap> g:= SL2Grading( L, sl2[2] );;
gap> g2:= Subspace( L, g[1][2] );;
gap> der:= LieDerivedSubalgebra(LieCentralizer(L,Subalgebra(L,[sl2[3]])));
<Lie algebra of dimension 33 over Rationals>
gap> BasisVectors( Basis( Intersection( g2, der ) ) );
[ v.18, v.23+(-1)*v.24+v.28, v.24+(-1)*v.25+(-1)*v.28,
v.27+(-1)*v.29+v.30+(-1)*v.31+(-1)*v.32, v.33+(-1)*v.36+v.37,
v.34+(-1)*v.36+v.37, v.39 ]
```

First we make some comments on the above computation. The \mathfrak{sl}_2 -triple comes ordered as (f, h, e) . So the second element is the neutral element, and the third element is the nil-positive element, i.e., the representative, which is as indicated above. So the second element defines the grading, which we compute with `SL2Grading`. In the subsequent line the subspace $\mathfrak{g}(2)$ is defined, followed by $[\mathfrak{g}_e, \mathfrak{g}_e]$. Finally a basis of the intersection is computed.

We see that one of the basis vectors of the intersection is

$$v.27+(-1)*v.29+v.30+(-1)*v.31+(-1)*v.32.$$

So we see that $e = (x_{29} + x_{32} + x_{31}) + (x_{27} + x_{30})$ does not lie in $[\mathfrak{g}_e, \mathfrak{g}_e]$ but $(x_{29} + x_{32} + x_{31}) - (x_{27} + x_{30})$ does!

3 The quotients \mathfrak{c}_e

Let \mathfrak{g} be a simple Lie algebra, and e a representative of a nilpotent orbit lying in an \mathfrak{sl}_2 -triple (f, h, e) . Note that the centraliser \mathfrak{g}_e is spanned by $\text{ad}h$ -eigenvectors with non-negative eigenvalues. We consider the quotient $\mathfrak{c}_e = \mathfrak{g}_e/[\mathfrak{g}_e, \mathfrak{g}_e]$, on which $\text{ad}h$ also acts. In Tables 7, 8, 9, 10, 11 we have listed the dimension of \mathfrak{c}_e for all nilpotent orbits of the Lie algebras of exceptional type, along with the eigenvalues of $\text{ad}h$ acting on \mathfrak{c}_e . Comparing these tables and the tables of [11] we get the following two results.

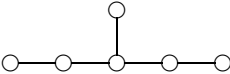
Proposition 2 *Let \mathfrak{g} be a simple Lie algebra of exceptional type. Let $e \in \mathfrak{g}$ be a representative of an induced nilpotent orbit lying in a unique sheet. Then the rank of that sheet is equal to the dimension of $\mathfrak{g}_e/[\mathfrak{g}_e, \mathfrak{g}_e]$, except the cases listed in Table 6.*

Table 6: Table of exceptions to Proposition 2.

\mathfrak{g}	label	weighted Dynkin diagram	rank	$\dim \mathfrak{g}_e/[\mathfrak{g}_e, \mathfrak{g}_e]$
E_6	$A_3 + A_1$	$\begin{array}{ccccc} & & 1 & & \\ & & & & \\ 0 & 1 & 0 & 1 & 0 \end{array}$	1	2
E_7	$D_6(a_2)$	$\begin{array}{ccccccc} & & 1 & & & & \\ & & & & & & \\ 0 & 1 & 0 & 1 & 0 & 2 & \end{array}$	2	3
E_8	$D_6(a_2)$	$\begin{array}{ccccccc} & & 1 & & & & \\ & & & & & & \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}$	1	3
E_8	$E_6(a_3) + A_1$	$\begin{array}{ccccccc} & & 0 & & & & \\ & & & & & & \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{array}$	1	3
E_8	$E_7(a_2)$	$\begin{array}{ccccccc} & & 1 & & & & \\ & & & & & & \\ 0 & 1 & 0 & 1 & 0 & 2 & 2 \end{array}$	3	4
F_4	$C_3(a_1)$	$\begin{array}{cccc} 1 & 0 & 1 & 0 \end{array}$	1	3

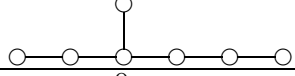
Proposition 3 *Let \mathfrak{g} be a simple Lie algebra of exceptional type, and let $e \in \mathfrak{g}$ be a nilpotent orbit that lies in more than one sheet. Then the maximal rank of such a sheet is strictly smaller than $\dim \mathfrak{g}_e/[\mathfrak{g}_e, \mathfrak{g}_e]$.*

Table 7: Nilpotent orbits in the Lie algebra of type E_6 .

label	weighted Dynkin diagram	$\dim \mathfrak{c}_e$	h -weights
			
A_1	$\begin{array}{ccccc} & & 1 & & \\ & & & & \\ 0 & 0 & 0 & 0 & 0 \end{array}$	0	
$2A_1$	$\begin{array}{ccccc} & & 0 & & \\ & & & & \\ 1 & 0 & 0 & 0 & 1 \end{array}$	1	0

<i>Nilpotent orbits in type E_6</i>						
$3A_1$	0	0	0	0	0	
A_2	0	0	2	0	0	2
$A_2 + A_1$	1	0	1	0	1	0
$2A_2$	2	0	0	0	2	2,4
$A_2 + 2A_1$	0	1	0	1	0	0
A_3	1	0	2	0	1	0, 2
$2A_2 + A_1$	1	0	0	0	1	
$A_3 + A_1$	0	1	1	1	0	0,2
$D_4(a_1)$	0	0	0	0	0	0,0,2,2,2
A_4	2	0	2	0	2	0,2,6
D_4	0	0	2	0	0	2,10
$A_4 + A_1$	1	1	1	1	1	0,2
A_5	2	1	0	1	2	2,4
$D_5(a_1)$	1	1	0	1	1	0,2,4
$E_6(a_3)$	2	0	2	0	2	2,2,2,4,4
D_5	2	0	2	0	2	0,2,6,10
$E_6(a_1)$	2	2	2	2	2	2,4,6,8,10
E_6	2	2	2	2	2	2,8,10,14,16,22

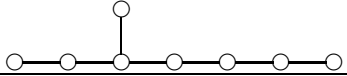
Table 8: Nilpotent orbits in the Lie algebra of type E_7 .

label	weighted Dynkin diagram	$\dim \mathfrak{c}_e$	h -weights
			
A_1	1 0 0 0 0 0 0	0	
$2A_1$	0 0 0 0 1 0 0	0	
$(3A_1)''$	0 0 0 0 0 2 0	1	2
$(3A_1)'$	0 1 0 0 0 0 0	0	

Nilpotent orbits in E_7 .								
A_2	2	0	0	0	0	0	1	2
$4A_1$	0	0	1	0	0	1	0	
$A_2 + A_1$	1	0	0	0	1	0	1	0
$A_2 + 2A_1$	0	0	0	1	0	0	0	
A_3	2	0	0	0	1	0	1	2
$2A_2$	0	0	0	0	2	0	1	2
$A_2 + 3A_1$	0	0	2	0	0	0	1	2
$(A_3 + A_1)''$	2	0	0	0	0	2	2	2,2
$2A_2 + A_1$	0	1	0	0	1	0	0	
$(A_3 + A_1)'$	1	0	0	1	0	0	1	2
$D_4(a_1)$	0	2	0	0	0	0	3	2,2,2
$A_3 + 2A_1$	1	0	0	1	0	1	1	2
D_4	2	2	0	0	0	0	2	2,10
$D_4(a_1) + A_1$	0	1	1	0	0	1	2	2,2
$A_3 + A_2$	0	0	0	1	0	1	2	0,2
A_4	2	0	0	0	2	0	3	0,2,6
$A_3 + A_2 + A_1$	0	0	0	2	0	0	1	2
$(A_5)''$	2	0	0	0	2	2	3	2,6,10
$D_4 + A_1$	2	1	1	0	0	1	1	2
$A_4 + A_1$	1	0	0	1	0	1	2	0,0
$D_5(a_1)$	2	0	0	1	0	1	3	0,2,4
$A_4 + A_2$	0	0	0	2	0	0	1	2
$(A_5)'$	1	0	0	1	0	2	1	2
$A_5 + A_1$	1	0	0	1	0	1	1	2
$D_5(a_1) + A_1$	2	0	0	0	2	0	2	2,2
$D_6(a_2)$	0	1	1	0	1	0	3	2,2,2

Nilpotent orbits in E_7 .								
$E_6(a_3)$	0	2	0	0	2	0	3	2,2,2
D_5	2	2	0	0	2	0	3	2,6,10
$E_7(a_5)$	0	0	2	0	0	2	6	2,2,2,2,2,2
A_6	0	0	2	0	2	0	2	2,10
$D_5 + A_1$	2	1	0	1	1	0	3	2,2,10
$D_6(a_1)$	2	1	0	1	0	2	4	2,2,6,10
$E_7(a_4)$	2	0	2	0	0	2	4	2,2,2,2
D_6	2	1	0	1	2	2	3	2,6,10
$E_6(a_1)$	2	0	2	0	2	0	5	0,2,4,6,10
E_6	2	2	2	0	2	0	4	2,10,14,22
$E_7(a_3)$	2	0	2	0	2	2	6	2,2,4,6,8,10
$E_7(a_2)$	2	2	0	2	0	2	5	2,2,6,10,14
$E_7(a_1)$	2	2	0	2	2	2	6	2,6,10,10,14,18
E_7	2	2	2	2	2	2	7	2,10,14,18,22,26,34

Table 9: Nilpotent orbits in the Lie algebra of type E_8 .

label	weighted Dynkin diagram							$\dim \mathfrak{c}_e$	h -weights
									
A_1	0	0	0	0	0	0	1	0	2
$2A_1$	1	0	0	0	0	0	0	0	
$3A_1$	0	0	0	0	0	1	0	0	
A_2	0	0	0	0	0	0	2	1	
$4A_1$	0	0	0	0	0	0	0	0	
$A_2 + A_1$	1	0	0	0	0	0	1	0	
$A_2 + 2A_1$	0	0	0	0	1	0	0	0	
A_3	1	0	0	0	0	0	2	1	2

Nilpotent orbits in E_8 .									
$A_2 + 3A_1$	0	1	0	0	0	0	0	0	
$2A_2$	2	0	0	0	0	0	0	1	2
$2A_2 + A_1$	1	0	0	0	0	1	0	0	
$A_3 + A_1$	0	0	0	0	1	0	1	1	2
$D_4(a_1)$	0	0	0	0	0	2	0	3	2,2,2
D_4	0	0	0	0	0	2	2	2	2,10
$2A_2 + 2A_1$	0	0	0	1	0	0	0	0	
$A_3 + 2A_1$	0	1	0	0	0	0	1	0	
$D_4(a_1) + A_1$	0	0	1	0	0	1	0	0	
$A_3 + A_2$	1	0	0	0	1	0	0	2	0,2
A_4	2	0	0	0	0	0	2	2	2,6
$A_3 + A_2 + A_1$	0	0	1	0	0	0	0	0	
$D_4 + A_1$	0	0	1	0	0	1	2	1	2
$D_4(a_1) + A_2$	0	0	2	0	0	0	0	1	2
$A_4 + A_1$	1	0	0	0	1	0	1	1	0
$2A_3$	1	0	0	1	0	0	0	0	
$D_5(a_1)$	1	0	0	0	1	0	2	2	2,4
$A_4 + 2A_1$	0	0	1	0	0	0	1	1	0
$A_4 + A_2$	0	0	0	0	2	0	0	1	2
A_5	2	0	0	0	1	0	1	1	2
$D_5(a_1) + A_1$	0	0	1	0	0	0	2	1	2
$A_4 + A_2 + A_1$	0	1	0	0	1	0	0	1	2
$D_4 + A_2$	0	0	2	0	0	0	2	2	2,2
$E_6(a_3)$	2	0	0	0	0	2	0	3	2,2,2
D_5	2	0	0	0	0	2	2	3	2,6,10
$A_4 + A_3$	0	0	1	0	0	1	0	0	


Nilpotent orbits in E_8 .									
$A_5 + A_1$	1	0	0	0	0	0	1	1	2
$D_5(a_1) + A_2$	0	1	0	0	1	0	1	1	2
$D_6(a_2)$	0	1	0	0	0	1	0	3	2,2,2
$E_6(a_3) + A_1$	1	0	0	1	0	1	0	3	2,2,2
$E_7(a_5)$	0	0	0	1	0	1	0	6	2,2,2,2,2,2
$D_5 + A_1$	1	0	0	1	0	1	2	2	2,10
$E_8(a_7)$	0	0	0	2	0	0	0	10	2,2,2,2,2,2,2,2,2
A_6	2	0	0	0	2	0	0	2	2,10
$D_6(a_1)$	0	1	0	0	0	1	2	3	2,2,10
$A_6 + A_1$	1	0	0	1	0	1	0	1	2
$E_7(a_4)$	0	0	0	1	0	1	2	3	2,2,2
$E_6(a_1)$	2	0	0	0	2	0	2	4	2,4,6,10
$D_5 + A_2$	0	0	0	2	0	0	2	3	0,2,2
D_6	2	1	0	0	0	1	2	2	2,6
E_6	2	0	0	0	2	2	2	4	2,10,14,22
$D_7(a_2)$	1	0	0	1	0	1	1	3	0,2,6
A_7	1	0	0	1	0	1	1	1	2
$E_6(a_1) + A_1$	1	0	0	1	0	1	0	3	0,2,4
$E_7(a_3)$	2	0	0	1	0	1	0	4	2,4,6,8
$E_8(b_6)$	0	0	2	0	0	0	2	5	2,2,2,2,4
$D_7(a_1)$	2	0	0	2	0	0	2	4	0,2,2,6
$E_6 + A_1$	1	0	0	1	0	1	2	2	2,10
$E_7(a_2)$	0	1	0	1	0	2	2	4	2,2,6,10
$E_8(a_6)$	0	0	2	0	0	2	0	6	2,2,2,6,6,6
D_7	2	1	0	1	1	0	1	2	2,10
$E_8(b_5)$	0	0	2	0	0	2	2	7	2,2,2,2,6,6,10

Nilpotent orbits in E_8 .									
$E_7(a_1)$	2	1	$\frac{1}{0}$	1	0	2	2	5	2,6,10,14,18
$E_8(a_5)$	2	0	$\frac{0}{2}$	0	0	2	0	5	2,2,2,10,10
$E_8(b_4)$	2	0	$\frac{0}{2}$	0	0	2	2	5	2,2,4,6,10
E_7	2	1	$\frac{1}{0}$	1	2	2	2	4	2,10,14,22
$E_8(a_4)$	2	0	$\frac{0}{2}$	0	2	0	2	6	2,4,6,8,10,14
$E_8(a_3)$	2	0	$\frac{0}{2}$	0	2	2	2	7	2,2,8,10,14,16,22
$E_8(a_2)$	2	2	$\frac{2}{0}$	2	0	2	2	6	2,6,10,14,18,22
$E_8(a_1)$	2	2	$\frac{2}{0}$	2	2	2	2	7	2,10,14,18,22,26,34
E_8	2	2	$\frac{2}{2}$	2	2	2	2	8	2,14,22,26,34,38,46,58

Table 10: Nilpotent orbits in the Lie algebra of type F_4 .

label	weighted Dynkin diagram	$\dim \mathfrak{c}_e$	h -weights
A_1	1 0 0 0	0	
\tilde{A}_1	0 0 0 1	0	
$A_1 + \tilde{A}_1$	0 1 0 0	0	
A_2	2 0 0 0	1	2
\tilde{A}_2	0 0 0 2	1	2
$A_2 + \tilde{A}_1$	0 0 1 0	0	
B_2	2 0 0 1	1	2
$\tilde{A}_2 + A_1$	0 1 0 1	1	2
$C_3(a_1)$	1 0 1 0	3	2,2,2
$F_4(a_3)$	0 2 0 0	6	2,2,2,2,2,2
B_3	2 2 0 0	2	2,10
C_3	1 0 1 2	2	2,10
$F_4(a_2)$	0 2 0 2	3	2,2,2
$F_4(a_1)$	2 2 0 2	4	2,4,6,10
F_4	2 2 2 2	4	2,10,14,22

Table 11: Nilpotent orbits in the Lie algebra of type G_2 .

label	weighted Dynkin diagram	$\dim \mathfrak{c}_e$	h -weights
			
A_1	1 0	1	2
\tilde{A}_1	0 1	0	
$G_2(a_1)$	2 0	3	2,2,2
G_2	2 2	2	2,10

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